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ON THE FORMULATION OF THE CONTACT PROBLEM OF ELASTIC PLASTICITY*

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A differential and a variational formulation of the problem of contact interaction between an elastic-plastic body and a rigid support are examined. Equations of the theory of plastic flow with isotropic hardening, which is a particular modification of the Il'yushin theory of elastic-plastic processes /1, 2/, are taken as governing relationships. A proof is presented of the existence and uniqueness of the generalized solution. To simplify the description the problem is considered in a Cartesian rectangular system of coordinates.

Contact problems with governing relationships of the deformation theory of plasticity are presented in /3/. Variational formulations utilizing generalized governing relationships of plasticity are formulated in /4, 5/. However, the constraints mentioned there on the generalized governing relationships are obviously inadequate for the uniqueness of the soluton.

1. *Differential formulation of the problem.* A quasistatic strain process is considered for an elastic-plastic body occupying a domain Ω in R^3 with a smooth boundary S. It is assumed that the displacements and the gradients of the displacements are small,and consequently, the squares of the gradients as well as the rotations of body elements can be neglected, and the connection to the side of the rigid support is considered to be ideal and unilateral. The problem is formulated in a reference system fixed with respect to the rigid support. The Mises plasticity condition is taken as the loading surface.

It is assumed that the domain under investgation Ω can consist of two parts at each instant: $\Omega^e = \{\mathbf{x} \in \Omega \mid \sigma_i(\mathbf{x}) < \sigma_T\}$ and $\Omega^p = \{\mathbf{x} \in \Omega \mid \sigma_i(\mathbf{x}) = \sigma_T\}$. Here σ_i is the stress intensity. Material strain occurs elastically in the domain Ω^e; in the general case the domain Ω^p consists of an active loading zone Ω^{pa} and an unloading zone Ω^{pr}, not known in advance and to be determined.

The conditions governing the above-mentioned zones have the form ($f = \sigma_i - \sigma_T$, $g_{ij} = \partial f/\partial\sigma_{ij}$):

$$\text{if } \mathbf{x} \in \Omega^p \text{ and } g_{ij}dS_{ij} \leqslant 0, \text{ then } \mathbf{x} \in \Omega^{pr},$$
$$\text{if } \mathbf{x} \in \Omega^p \text{ and } g_{ij}dS_{ij} > 0, \text{ then } \mathbf{x} \in \Omega^{pa}.$$

We write the governing relationships in the domain Ω as

$$dS_{ij} = 2G\,(de_{ij} - d\lambda\, g_{ij}), \quad d\sigma = K\, d\varepsilon \tag{1.1}$$

where $d\sigma$, $d\varepsilon$ are increments of the mean pressure and the mean strain. The scalar factor $d\lambda$ equals zero in the domains Ω^e and Ω^{pr}. In the domain Ω^{pa}

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$$d\lambda = E_a^{-1} g_{ij}\, dS_{ij} \tag{1.2}$$

It is assumed that the tangential modulus E_a /6/ satisfies the conditions

$$\alpha \geqslant E_a \geqslant k > 0, \quad E_a \in L_\infty(\Omega^p) \tag{1.3}$$

Taking relationship (1.2) as well as the conditions governing the zones Ω^{pr} and Ω^{pa} into account, we write the governing relationships for the factor $d\lambda$ in the form /7/

$$d\lambda = 0 \text{ in } \Omega^e \tag{1.4}$$

$$d\lambda \geqslant 0, \quad E_a\, d\lambda - g_{ij}\, dS_{ij} \geqslant 0$$

$$d\lambda\,(E_a\, d\lambda - g_{ij}\, dS_{ij}) = 0 \text{ in } \Omega^p \tag{1.5}$$

We will now describe the boundary conditions. In the general case the boundary S consists of three parts $S = S_\sigma \cup S_u \cup S_c$. The surface S_c includes the actual contact surface and the zone of possible contact.

To determine the part S_c we assume that the boundary of the absolutely rigid support is described by the equation $\Psi(\mathbf{x}) = 0$, where $\Psi(\mathbf{x}) < 0$ within the stamp and $\Psi(\mathbf{x}) > 0$ outside.

Henceforth, small displacements are considered throughout, and consequently, it is natural to refer the points S lying on the surface of the rigid support (these points form the actual contact surface S_k) and the points S sufficiently closely located to the support, to S_c. This can be done by defining equidistant surfaces by the equation $\Psi(\mathbf{x}) = C\ (C > 0)$ then $S_c = \{\mathbf{x} \in S \mid 0 \leqslant \Psi(\mathbf{x}) \leqslant \varepsilon_c\}$. Here ε_c is a fixed positive parameter found from physical considerations. Using the function $\Psi(\mathbf{x})$ we determine the surface of actual contact $S_k = \{\mathbf{x} \in S_c \mid \Psi(\mathbf{x}) = 0\}$ and the domain of possible contact $S^0 = \{\mathbf{x} \in S_c \mid \times \mid 0 < \Psi(\mathbf{x}) \leqslant \varepsilon_c\}$.

We assume that the body is not subjected to the action of surface forces on the surface S^0, i.e., $\sigma_{ij} n_j = 0$ (n_j are components of the unit vector of the external normal to S).

Using the results in /8/, we write the condition of non-penetration on the surface S_c in linearized form with respect to the desired function

$$du_\nu \leqslant \delta$$

$$du_\nu = \boldsymbol{\nu}\, d\mathbf{u}, \quad \boldsymbol{\nu} = -\operatorname{grad}\Psi(\mathbf{x})/|\operatorname{grad}\Psi(\mathbf{x})|, \quad \delta = \Psi(\mathbf{x})/|\operatorname{grad}\Psi(\mathbf{x})|$$

The quantity δ defines the gap in the zone of possible contact $S^0\ (\delta > 0)$ and in the zone of actual contact $\delta = 0$.

Taking the above into account, we write the equations and boundary conditions of the problem in the form

$$d\sigma_{ij,\,j} + dF_i = 0 \text{ in } \Omega \tag{1.6}$$

$$d\sigma_{ij} = dS_{ij} + \delta_{ij}\, d\sigma, \quad d\sigma = {}^1/_3 K \delta_{ij}\, dE_{ij}, \quad dS_{ij} = 2G\,(de_{ij} - g_{ij}\, d\lambda)$$

$$de_{ij} = dE_{ij} - {}^1/_3 \delta_{ij}\delta_{kl}\, dE_{kl}, \quad dE_{ij} = {}^1/_2\,(du_{i,\,j} + du_{j,\,i}) \text{ in } \Omega \tag{1.7}$$

$$du_i = 0 \text{ on } S_u, \quad d\sigma_{ij} n_j = dP_i \text{ on } S_\sigma \tag{1.8}$$

$$du_\nu \leqslant \delta, \quad du_\nu < \delta \Rightarrow \sigma_\nu{}^t + d\sigma_\nu = 0, \quad du_\nu = \delta \Rightarrow \sigma_\nu{}^t + d\sigma_\nu \leqslant 0$$

$$d\boldsymbol{\sigma}_T = \mathbf{0} \text{ on } S_c \tag{1.9}$$

$$(d\sigma_\nu = \boldsymbol{\nu}\, d\boldsymbol{\sigma}\boldsymbol{\nu}, \quad d\boldsymbol{\sigma}_T = d\boldsymbol{\sigma}\boldsymbol{\nu} - d\sigma_\nu \boldsymbol{\nu})$$

Appended to these relationships are (1.4) and (1.5) written for the domain Ω^p.

Here $d\sigma$ is the stress tensor increment, $\sigma_\nu{}^t = \boldsymbol{\nu}\boldsymbol{\sigma}^t\boldsymbol{\nu}$ is the reaction of the rigid support, and $\boldsymbol{\sigma}^t$ is the stress tensor at the time t. In the zone of possible contact $\sigma_\nu{}^t = 0$ while $\sigma_\nu{}^t \leqslant 0$ in the zone of actual contact. The state of the body at the initial instant is considered to be unstressed and unstrained, assuming in this case that $\mathbf{u}|_{t=0} = \mathbf{0}$.

Let us determine the sets

$$M = \{d\mathbf{v} \in (C^2(\bar{\Omega}))^3 \mid dv_i = 0 \text{ on } S_u,\ dv_\nu \leqslant \delta \text{ on } S_c\}$$

$$Q = \{d\mu \in L_2(\Omega) \mid d\mu \geqslant 0 \text{ in } \Omega^p,\ d\mu = 0 \text{ in } \Omega^e\}$$

We will call the pair of functions $(d\mathbf{u}, d\lambda) \in M \times Q$, satisfying the equations and boundary conditions (1.4)-(1.9) the classical solution.

We note that the relationships (1.4)-(1.9) enable us to connect the displacement $\mathbf{u}^t$, strain $\mathbf{E}^t$, and stress $\boldsymbol{\sigma}^t$ fields at the time t to the appropriate fields $\mathbf{u}^{t+dt} = \mathbf{u}^t + d\mathbf{u}$, $\mathbf{E}^{t+dt} = \mathbf{E}^t + d\mathbf{E}$, $\boldsymbol{\sigma}^{t+dt} = \boldsymbol{\sigma}^t + d\boldsymbol{\sigma}$ at the time $t + dt$.

2. *Variational formulation.* Let $(d\mathbf{u}, d\lambda)$ be the classical solution. We multiply the equilibrium Eq.(1.6) scalarly by $d\mathbf{v} - d\mathbf{u}$, where $d\mathbf{v} \in M$, and we integrate over the domain Ω. Then applying Gauss's theorem to the expression obtained and taking account of the boundary conditions (1.8) and (1.9) and the first three governing relationships (1.7), we can obtain the inequality

$$a(d\mathbf{u}, d\mathbf{v}-d\mathbf{u}) - \int_\Omega 2Gg_{ij}\, d\lambda\, de_{ij}(d\mathbf{v}-d\mathbf{u})\, d\Omega - \langle \mathbf{f}, d\mathbf{v}-d\mathbf{u}\rangle \geqslant 0, \tag{2.1}$$

$$\forall\, d\mathbf{v} \in M \tag{2.2}$$

$$a(d\mathbf{u}, d\mathbf{v}) = \int_\Omega 2G\, de_{ij}(d\mathbf{u})\, de_{ij}(d\mathbf{v})\, d\Omega + \int_\Omega 3\, d\sigma(d\mathbf{u})\, dE(d\mathbf{v})\, d\Omega \tag{2.3}$$

$$\langle \mathbf{f}, d\mathbf{v}\rangle = \int_\Omega dF_i\, dv_i\, d\Omega + \int_{S_\sigma} dP_i\, dv_i\, dS_\sigma - \int_{S_c} \sigma_\nu^t\, dv_\nu\, dS_c$$

Using relationships (1.4) and (1.5), we form another inequality

$$\int_\Omega (d\mu - d\lambda)(E_a\, d\lambda - g_{ij}\, dS_{ij}(d\mathbf{u}))\, d\Omega \geqslant 0, \quad \forall\, d\mu \in Q \tag{2.4}$$

We introduce the notation

$$dv = (d\mathbf{v},\ d\mu) \in M \times Q,\ du = (d\mathbf{u},\ d\lambda) \in M \times Q$$

$$b(d\mu, d\lambda) = \int_\Omega d\mu\, d\lambda\, (E_a + 2Gg_{ij}g_{ij})\, d\Omega$$

$$c(d\mu, d\mathbf{v}) = \int_\Omega d\mu\, de_{ij}(d\mathbf{v})\, 2Gg_{ij}\, d\Omega$$

$$A(du,\ dv) = a(d\mathbf{u},\ d\mathbf{v}) + b(d\mu,\ d\lambda) - c(d\mu,\ d\mathbf{u}) - c(d\lambda,\ d\mathbf{v})$$

Combining the inequalities (2.1) and (2.4) and using the notation assumed above, we obtain

$$A(du,\ dv - du) - \langle \mathbf{f},\ d\mathbf{v} - d\mathbf{u}\rangle \geqslant 0,\ \forall dv \in M \times Q \tag{2.5}$$

Let us examine the following problem: it is required to determine

$$du \in M \times Q : A(du,\ dv - du) - \langle \mathbf{f},\ d\mathbf{v} - d\mathbf{u}\rangle \geqslant 0,\ \forall dv \in M \times Q \tag{2.6}$$

It follows from the considerations presented above that if a classical solution of the problem exists, then it satisfies the variational inequality (2.6).

The converse also holds. Let du be a solution of problem (2.6). Then du is a classical solution.

The proof is as follows.

Setting $d\mu = d\lambda$, we obtain inequality (2.1) from (2.6). Using geometrical and physical relationships (1.7) (assuming here that the solution du satisfies it exactly), it can be established from inequality (2.1) that the solution du satisfies the equilibrium Eqs.(1.6) and the boundary conditions (1.8) and (1.9).

Now we take $d\mathbf{v} = d\mathbf{u}$. Then taking account of the third governing relationship (1.7), inequality (2.6) takes the form (2.4). We have $d\lambda = 0$ in the domain Ω^e by the definition of the set Q. By virtue of the arbitrariness of the selection of du and the definition of the set Q, relationships (1.4) and (1.5) follow in the domain Ω^p from the inequality (2.4).

Let us introduce the generalized solution of problem (2.6) into the consideration. We understand this latter solution to be the solution of the variational inequality (2.6) in a broader class of functions $V \times Q$ where

$$V = \{d\mathbf{v} \in (H^1(\Omega))^3,\ dv_i = 0 \text{ on } S_u,\ dv_\nu \leqslant \delta \text{ on } S_c\}$$

$(H^1(\Omega))^3$ is a Sobolev vector space, $S \in C^1$.

The proof of the existence and uniqueness of the generalized solution requires verification of the conditions of the following theorem /9/.

Theorem. Let U be a Hilbert space, $A(dv, du)$ a coercive bilinear continuous form on U; $V \times Q \subset U$ is a closed convex set, $\mathbf{f} \in U^*$. Then a unique generalized solution of problem (2.6) exists.

We introduce the space $(Z(\Omega))^3 = \{d\mathbf{v} \in (H^1(\Omega))^3 \mid dv_i = 0$ on $S_u\}$, $U = (Z(\Omega))^3 \times L_2(\Omega)$. Norms in the spaces $L_2(\Omega)$, $(Z(\Omega))^3$ /10/ and U

$$\| d\mu \|_{L_2} = \Big(\int_\Omega (d\mu)^2\, d\Omega\Big)^{1/2}, \quad \| d\mathbf{v} \|_Z = \Big(\int_\Omega d\mathbf{E}_{ij}(d\mathbf{v})\, d\mathbf{E}_{ij}(d\mathbf{u})\, d\Omega\Big)^{1/2}$$

$$\| dv \|_U = (\| d\mathbf{v} \|_Z^2 + \| d\mu \|_{L_2}^2)^{1/2}$$

and required below.

The form $A(dv, du)$ is linear in each argument. Let us confirm the boundedness property. It follows from relationship (2.2) that

$$|a(d\mathbf{v}, d\mathbf{u})| \leqslant \beta \|d\mathbf{u}\|_Z \|d\mathbf{v}\|_Z, \quad \forall\, d\mathbf{u}, d\mathbf{v} \in (Z(\Omega))^3, \quad \beta = E/(1-2\nu) \tag{2.7}$$

where E is Young's modulus, and ν is Poisson's ratio.

After calculating the derivatives of the loading function, the inequality

$$\begin{gathered} |b(d\mu, d\lambda)| \leqslant \beta_1 \|d\mu\|_{L_2} \|d\lambda\|_{L_2}, \ \forall d\lambda, d\mu \in L_2 \\ \beta_1 = \max E_a + 3G = \alpha + 3G \end{gathered} \tag{2.8}$$

can be established for the mapping of $b(d\mu, d\lambda)$.

Now we prove the boundedness of the mapping $c(d\mu, d\mathbf{v})$.

Applying the Cauchy inequality and performing a chain of manipulations, we obtain

$$\begin{gathered} |c(d\mu, d\mathbf{v})| \leqslant \|d\mu\|_{L_2} \Big(\int\limits_\Omega [e_{ij}(d\mathbf{v})\, 2Gg_{ij}]^2\, d\Omega\Big)^{1/2} \leqslant \beta_2 \|d\mu\|_{L_2} \|d\mathbf{v}\|_Z \\ \forall\, d\mu \in L_2, \quad \forall\, d\mathbf{v} \in (Z(\Omega))^3, \quad \beta_2 = G\sqrt{6} \end{gathered} \tag{2.9}$$

The inequalities (2.7), (2.8) and (2.9) enable us to obtain the estimate

$$\begin{gathered} |A(dv, du)| \leqslant 2\beta_3 \|du\|_U \|dv\|_U, \ \forall du, dv \in U \\ \beta_3 = \max(\beta, \beta_1, \beta_2) \end{gathered} \tag{2.10}$$

Therefore, $A(dv, du)$ is a bilinear bounded and, consequently, continuous form on U.

Let us confirm the coercivity condition. The inequality

$$a(d\mathbf{v}, d\mathbf{v}) \geqslant \beta_4 \|d\mathbf{v}\|_Z^2, \ \forall\, d\mathbf{v} \in (Z(\Omega))^3, \ \beta_4 = E/(1+\nu) = 2G \tag{2.11}$$

holds for the mapping $a(d\mathbf{v}, d\mathbf{v})$.

Using inequalities (2.11) and (2.9) and the second inequality in (1.3), we perform a chain of manipulations

$$\begin{gathered} A(dv, dv) \geqslant 2G\|d\mathbf{v}\|_Z^2 + (k+3G)\|d\mu\|_{L_2}^2 - G\sqrt{6}\|d\mu\|_{L_2}\|d\mathbf{v}\|_Z \geqslant \\ 2G\|d\mathbf{v}\|_Z^2 + (k+3G)\|d\mu\|_{L_2}^2 - G(3+\beta^*)\|d\mu\|_{L_2}^2 - \\ G(2 - {}^1/_2\beta^*)\|d\mathbf{v}\|_Z^2 = {}^1/_2 G\beta^*\|d\mathbf{v}\|_Z^2 + (k - G\beta^*)\|d\mu\|_{L_2}^2, \\ \forall\, dv \in u, \quad \forall \beta^* \in [0, 1] \end{gathered}$$

We take $\beta^* = 2k/(3G)$. Then

$$A(dv, dv) \geqslant \beta_5 \|dv\|_U^2, \quad \forall dv \in U, \ \beta_5 = {}^1/_3 k$$

We require from the bulk and surface forces that $\mathbf{f} \in U^*$. Therefore, all the conditions of the theorem are satisfied and problem (2.6) has a unique, generalized solution.

It should be noted that the numerical solution of problem (2.6) can be found by the finite element method in combination with the relaxation method.

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SLIP LINES AT THE CORNER OF THE INTERFACIAL BOUNDARY OF DIFFERENT MEDIA*

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A symmetric problem on the initial development of a plastic wave, simulated by two straight slip lines starting from the apex, near the corner of the interfacial boundary of different media is examined under plane strain conditions. The exact analytical solution is constructed for the Wiener-Hopf functional equation of the problem. A formula is deduced to determine the slip line length, and their slope to the interfacial boundary of the media is established.

1. We consider the problem of the initial development of the plastic zone near a corner O of the interfacial boundary of media (Fig.1) under plane strain conditions in a domain consisting of two homogeneous isotropic parts *1* and *2* with Young's moduli and Poisson's ratios E_1, ν_1 and E_2, ν_2, respectively. The problem is assumed to be symmetrical about the bisectrix of the corner. It is assumed that the plastic strains are concentrated along two straight slip lines starting from the apex, whose length is small compared with the body dimensions.

Using the "microscope principle", we arrive at a plane static symmetric problem of elasticity theory of the class N /1/ for a piecewise-homogeneous plane with interfacial boundary of the media in the form of the sides $\theta = \beta$ and $\theta = \beta - 2\alpha$ ($\alpha \in]0;\ \pi/2\ [\ \bigcup\ ]\ \pi/2;\ \pi[$) of an angle containing slip lines for $\theta = 0,\ r < l$ and for $\theta = 2(\beta - \alpha),\ r < l$. An asymptotic form is realized at infinity that is the greatest solution, at infinity, of an analogous problem for a piecewise-homogeneous plane without slip lines that satisfies the stress decay condition at infinity. The latter is constructed by the method of singular solutions /1/ and is determined apart from an arbitrary constant C. This constant, that characterizes the external field strength, is considered given. It is found from the solution of the external problem.

It is required to determine the slip line length l and the angle β of their slope to the interfacial boundary of the media.

Confining ourselves to an examination of the half-plane $\beta - \alpha \leqslant \theta \leqslant \pi - \alpha + \beta$, we write the boundary conditions thus:

$$\theta = \beta,\ \langle\sigma_\theta\rangle = \langle\tau_{r\theta}\rangle = 0,\ \langle u_\theta\rangle = \langle u_r\rangle = 0 \tag{1.1}$$

$$\theta = \beta - \alpha,\ \theta = \pi - \alpha + \beta,\ \tau_{r\theta} = 0,\ u_\theta = 0$$

$$\theta = 0,\ \langle\sigma_\theta\rangle = \langle\tau_{r\theta}\rangle = 0,\ \langle u_\theta\rangle = 0 \tag{1.2}$$

$$\theta = 0,\ r < l,\ \tau_{r\theta} = \tau_1;\ \theta = 0,\ r > l,\ \langle u_r\rangle = 0$$

$$\theta = 0,\quad r \to l + 0,\quad \tau_{r\theta} \sim \frac{k_{II}}{\sqrt{2\pi(r - l)}} \tag{1.3}$$

$$\theta = 0,\quad r \to l - 0,\quad \left\langle\frac{\partial u_r}{\partial r}\right\rangle \sim -\frac{4(1 - \nu_1^2)}{E_1}\frac{k_{II}}{\sqrt{2\pi(l - r)}}$$

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